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TWO TRANSFORMS OF PLANE CURVES AND THEIR FUNDAMENTAL GROUPS

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§1. Introduction. Let $C = \{(X;Y;Z) \in F(X,Y,Z) = 0\}$ be a projective curve and let $C^a = \{f(x,y) = 0\} \subset \mathbf{C}^2$ be the corresponding affine plane curve with respect to the affine coordinate space $\mathbf{C}^2 = \mathbf{P}^2 - \{Z = 0\}$, $x = X/Z$, $y = Y/Z$ and $f(x,y) = F(x,y,1)$. In this paper, we study two basic operations. First we consider an n-fold cyclic covering $\varphi_n : \mathbf{C}^2 \to \mathbf{C}^2$, $\varphi_n(x,y) = (x, (y-\beta)^n + \beta)$, branched along a line $D = \{y = \beta\}$ for an arbitaray positive integer $n \geq 2$. Let $\mathcal{C}_n(C;D)$ be the projective closure of the pull back $\varphi_n^{-1}(C^a)$ of C^a. The behavior of φ_n at infinity gives an interesting effect on the fundamental group. In our previous paper [O6], we have studied the double covering φ_2 to construct some interesting plane curves, such as a Zariski's three cuspidal quartic and a conical six cuspidal sextic.

Secondly we consider the following Jung transform of degree n, $J_n : \mathbf{C}^n \to \mathbf{C}^n$, $J_n(x,y) = (x + y^n, y)$ and let $\mathcal{J}_n(C; L_\infty)$ be the projective compactification of $J_n^{-1}(C^a)$. Though J_n is a automorphism of $\mathbf{C}^2$, the behavior of J_n or $\mathcal{J}_n(C)$ at infinity is quite interesting.

Both of φ_n and J_n can be extended canonically to rational mapping from $\mathbf{P}^2$ to $\mathbf{P}^2$ and they are not defined only at $[1;0;0]$ and constant along the line at infinity $L_\infty = \{Z = 0\}$. They have also the following similarity. For a generic φ_n and a generic J_n, there exist surjective homomorphisms

$$\Phi_n : \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C)) \to \pi_1(\mathbf{P}^2 - C), \quad \Psi_n : \pi_1(\mathbf{P}^2 - \mathcal{J}_n(C)) \to \pi_1(\mathbf{P}^2 - C)$$

and both kernels $\operatorname{Ker}\Phi_n$ and $\operatorname{Ker}\Psi_n$ are cyclic group of order n which are subgroups of the respective centers of $\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C))$ and $\pi_1(\mathbf{P}^2 - \mathcal{J}_n(C))$ (Theorem (3.5) and Theorem (4.3)).

Both operations are useful to construct examples of interesting plane curves, starting from a simple plane curve. Applying this operation to a Zariski's three cuspidal quartic Z_4, we obtain new examples of plane curves $\mathcal{C}_n(Z_4)$ and $\mathcal{J}_n(Z_4)$ of degree $4n$ whose complement in $\mathbf{P}^2$ has a non-commutative finite fundamental group of order $12n$ (§5). We will construct a new example of Zariski pair $\{\mathcal{C}_3(Z_4), C_2\}$ of curves of degree 12 (§5).

In §6, we study non-atypical curves and their Jung transforms. We use a non-generic Jung transform to construct a rational curve $\widetilde{C}$ of degree pq for any p, q with $\gcd(p,q) = 1$ such that $\widetilde{C}$ has two irreducible singularities and the fundamental $\pi_1(\mathbf{P}^2 - \widetilde{C})$ is isomorphic to the free product $\mathbf{Z}/p\mathbf{Z} * \mathbf{Z}/q\mathbf{Z}$ (Corollary (6.6.1)). This paper is composed as follows.

§2. Basic properties of $\pi_1(\mathbf{P}^2 - C)$ and Zariski's pencil method.
§3. Cyclic transforms of plane curves.
§4. Jung transforms of plane curves.
§5. Zariski's quartic and Zariski pairs
§6. Non-atypical curves and some examples.

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§2. Basic properties of $\pi_1(\mathbf{P}^2 - \mathbf{C})$ and Zariski's pencil method. Let C be a reduced projective curve of degree d and let $C_1, \dots, C_r$ be the irreducible components of C and let d_i be the degree of C_i. So $d = d_1 + \cdots + d_r$. First we recall that the first homology of the complement is given by the Lefschetz duality and by the exact sequence of the pair $(\mathbf{P}^2, C)$ as follows.

$$(2.1) \qquad H_1(\mathbf{P}^2 - C) \cong \mathbf{Z}^r/(d_1, \dots, d_r) \cong \mathbf{Z}^{r-1} \oplus \mathbf{Z}/d_0\mathbf{Z}$$

where $d_0 = \gcd(d_1, \dots, d_r)$ and $\mathbf{Z}^r = \mathbf{Z}\oplus\cdots\oplus\mathbf{Z}$ (r factors). In particular, if C is irreducible ($r = 1$), we have $H_1(\mathbf{P}^2 - C) \cong \mathbf{Z}/d\mathbf{Z}$ and $H_1(\mathbf{C}^2 - C^a) \cong \mathbf{Z}$ where $\mathbf{C}^2 := \mathbf{P}^2 - L_\infty$ and $C^a := C \cap L_\infty$.

(2.2) van Kampen-Zariski's pencil method. We fix a point $B_0 \in \mathbf{P}^2$ and we consider the pencil of lines $\{L_\eta,\ \eta \in \mathbf{P}^1\}$ through B_0. Taking a linear change of coordinates if necessary, we may assume that L_η is defined by $L_\eta = \{X - \eta Z = 0\}$ and $B_0 = [0;1;0]$ in homogeneous coordinates. Take $L_\infty = \{Z = 0\}$ as the line at infinity and we write $\mathbf{C}^2 = \mathbf{P}^2 - L_\infty$. Note that $L_\infty = \lim_{\eta\to\infty} L_\eta$. We assume that $L_\infty \not\subset C$. We consider the affine coordinates $(x,y) = (X/Z, Y/Z)$ on $\mathbf{C}^2$ and let $F(X,Y,Z)$ be the defining homogeneous polynomial of C and let $f(x,y) := F(x,y,1)$ be the affine equation of C. In this affine coordinates, the pencil line L_η is simply defined by $\{x = \eta\}$. As we consider two fundamental groups $\pi_1(\mathbf{P}^2 - C)$ and $\pi_1(\mathbf{P}^2 - C \cup L_\infty)$ simultaneously, we use the notations : $C^a = C \cap \mathbf{C}^2$ and $L_\eta^a = L_\eta \cap \mathbf{C}^2 \cong \mathbf{C}$. We identify hereafter L_η and L_η^a with $\mathbf{P}^1$ and $\mathbf{C}$ respectively by $y : L_\eta \cong \mathbf{P}^1$ for $\eta \neq \infty$. Note that the base point of the pencil B_0 corresponds to $\infty \in \mathbf{P}^1$.

We say that the pencil $L_\eta = \{x = \eta\}$, $\eta \in \mathbf{C}$, is *admissible* if there exists an integer $d' \le d$ which is independent of $\eta \in \mathbf{C}$ such that $C^a \cap L_\eta^a$ consists of d' points counting the multiplicity. This is equivalent to : *$f(x,y)$ has degree d' in y and the coefficient of $y^{d'}$ is a non-zero constant.* Note that if $B_0 \notin C$, L_η is admissible and $d' = d$. If $d' < d$, $B_0 \in C$ and the intersection multiplicity $I(C, L_\infty; B_0) = d - d'$.

Proposition (2.2.2). (1) *The canonical homomorphism $j_\sharp : \pi_1(L_{\eta_0}^a - L_{\eta_0}^a \cap C^a; b_0) \to \pi_1(\mathbf{C}^2 - C^a; b_0)$ is surjective and the kernel* $\operatorname{Ker} j_\sharp$ *is equal to $\mathcal{M}$ and therefore $\pi_1(\mathbf{C}^2 - C^a; b_0)$ is isomorphic to the quotient group $G/\mathcal{M}$.*
(2) *The canonical homomorphism $\iota_\sharp : \pi_1(\mathbf{C}^2 - C^a; b_0) \to \pi_1(\mathbf{P}^2 - C; b_0)$ is surjective. If $B_0 \notin C$ (so $d' = d$), the kernel* $\operatorname{Ker}\iota_\sharp$ *is normally generated by $\omega = g_d \cdots g_1$.*

Assume further that $B_0 \notin C$ and L_∞ is generic. Then
(3) ([O3]) *ω is in the center of $\pi_1(\mathbf{C}^2 - C^a)$. Therefore* $\operatorname{Ker}(\iota_\sharp) = \langle[\omega]\rangle \cong \mathbf{Z}$.
(4) *$\iota_\sharp$ induces an isomorphism of the commutator groups: $\iota_{\sharp\mathcal{D}} : \mathcal{D}(\pi_1(\mathbf{C}^2 - C^a)) \xrightarrow{\cong} \mathcal{D}(\pi_1(\mathbf{P}^2 - C))$ and an exact sequence of first homologies: $0 \to \langle[\omega]\rangle \cong \mathbf{Z} \to H_1(\mathbf{C}^2 - C) \to H_1(\mathbf{P}^2 - C) \to 0$.*

Proof. The assertions are well-known except (4). So we only need to show the assertion (4). First $\iota_{\sharp\mathcal{D}}$ is surjective. As the homology class $[\omega]$ of ω is given by $[(0, d_1, \dots, d_r)]$ under the identification $H_1(\mathbf{C}^2 - C^a) \cong \mathbf{Z}^{r+1}/(1, d_1, \dots, d_r)$, $[\omega]$ generates an infinite cyclic group. Thus the injectivity of $\iota_{\sharp\mathcal{D}}$ follows from $\mathcal{D}(\pi_1(\mathbf{P}^2 - C)) \cap \operatorname{Ker}\iota_\sharp = \{e\}$. The second exact sequence follows from the first isomorphism and the property: $\langle\omega\rangle \cap \mathcal{D}(\pi_1(\mathbf{C}^2 - C^a)) = \{e\}$. $\square$

We usually denote $G/\mathcal{M}$ as $\pi_1(\mathbf{C}^2 - C^a; b_0) = \langle g_1, \dots, g_d; R(\sigma_1), \dots, R(\sigma_\ell)\rangle$. We call $\pi_1(\mathbf{C}^2 - C^a)$ *the fundamental group of a generic affine complement of C* if L_∞ is generic. Note that if L_∞ is generic, $\pi_1(\mathbf{C}^2 - C^a)$ does not depend on the choice of a line at infinity L_∞.

(2.3) Bracelets and lassos. An element $\rho \in \pi_1(\mathbf{P}^2 - C; b_0)$ is called *a lasso* for C_i if it is represented by a loop $\mathcal{L} \circ \tau \circ \mathcal{L}^{-1}$ where τ is a counter-clockwise oriented boundary of a small normal disk $D_i(P)$ of C_i at a regular point $P \in C_i$ such that $D_i(P) \cap (C \cup L_\infty) = \{P\}$ and $\mathcal{L}$ is a path connecting b_0 and τ. We call τ *a bracelet* for C_i. It is easy to see that any two bracelets τ and τ' for the same irreducible component, say C_i, are free homotopic. Therefore *the homotopy class of a lasso for C_i (or L_∞) is unique up to a conjugation.* We say that the line at infinity L_∞ is *central* for C if there is a lasso ω for L_∞ which is in the center of $\pi_1(\mathbf{C}^2 - C^a) = \pi_1(\mathbf{P}^2 - C \cup L_\infty)$. If L_∞ is generic for C, L_∞ is central by Proposition (2.2.2) but the converse is not always true (see Corollary (3.3.1) and Theorem (4.3)).

Assume that L_∞ is central for C and take an admissible pencil $\{L_\eta, \eta \in \mathbf{C}\}$ with the base point $B_0 \notin C$. Then ω is in the center of $\pi_1(\mathbf{C}^2 - C^a; b_0)$. Thus we can replace the homotopy deformation of ω by free homotopy deformation of Ω. This viewpoint is quite useful in the later sections.

Remark (2.4). Suppose that $B_0 \notin C$ and L_∞ is *not generic*. Take $\Delta = \{\eta \in \mathbf{C}_B; |\eta| \le R\} \subset \mathbf{C}_B$ as before and we may assume that $\eta_0 \in \partial\Delta$ and let $\sigma_\infty := \partial\Delta$. The monodromy relation $g_i^{-1} g_i^{\sigma_\infty}$ is contained in the group of monodromy relations $\mathcal{M}$. We can also consider the monodromy relation around $\eta = \infty$. For this purpose, we identify $L_\eta \cong \mathbf{P}^1$ through another rational function $\varphi := Y/X$ for $|\eta| \ge R$. For $\eta \ne 0$, $\varphi : L_\eta \to \mathbf{C}$ is written as $\varphi(\eta, y) = y/\eta$. Let $j_\theta : L_{\eta_0} \to L_{\eta_0 \exp(\theta i)}$, $0 \le \theta \le 2\pi$ be a family of homeomorphisms which is identity outside of a big disk under this identification $\varphi : L_\eta \to \mathbf{C}$. Then the base point b_0 stays constant under the identification by φ but under the first identification of $y : L_\eta \to \mathbf{P}^1$, this gives a rotation: $\theta \mapsto b_0 \exp(\theta i)$. Putting $h' = j_{2\pi}$, this implies that the monodromy relation around L_∞ is given by

$$[h'_\sharp(g)] = \omega g^{-\sigma_\infty} \omega^{-1}, \quad g \in G \tag{2.4.1}$$

This gives the following corollary.

Corollary (2.4.2). *Take another generic line $L_{\eta'_0}$ for C with $\eta'_0 \ne \eta_0$. Let $R_1, \dots, R_\ell$ be the monodromy relation along σ_i as before. Then the fundamental group of a generic affine complement $\pi_1(\mathbf{P}^2 - C \cup L_{\eta'_0}; b_0)$ is isomorphic to the quotient group of $\pi_1(\mathbf{C}^a - C^a; b_0)$ by the relation $\omega g_i = g_i \omega$, $i = 1, \dots, d$. In particular, if ω is in the center of $\pi_1(\mathbf{C}^2 - C^a; b_0)$, $\pi_1(\mathbf{C}^2 - C^a; b_0)$ is isomorphic to the fundamental group of a generic affine complement $\pi_1(\mathbf{P}^2 - C \cup L_{\eta'_0}; b_0)$.*

Proof. Changing coordinates if necessary, we may assume that $\eta'_0 = 0$. Using the second identification $Y/X : L_\eta \cong \mathbf{P}^1$ for $\eta \ne 0$, we can write the monodromy relation $R(\infty)$ at $\eta = \infty$ as $R(\infty) : \ g_j = [h'_\sharp(g_j)]$, for $j = 1, \dots, d$ and the other monodromy relations $R_i, i = 1, \dots, \ell$ are the same with those which are obtained from the first identification. Therefore we have $\pi_1(\mathbf{P}^2 - C \cup L_{\eta'_0}; b_0) \cong \langle g_1, \dots, g_d; R_1, \dots, R_\ell, R(\infty) \rangle$. On the other hand, we know that $\omega = g_d \cdots g_1$ is in the center of $\pi_1(\mathbf{P}^2 - C \cup L_{\eta'_0}; b_0)$ ([O2]). Thus we get $(\star) : \ \omega g_j = g_j \omega$, $j = 1, \dots, d$ in $\pi_1(\mathbf{P}^2 - C \cup L_{\eta'_0}; b_0)$. Conversely in the group $\langle g_1, \dots, g_d; R_1, \dots, R_\ell, (\star) \rangle$, we have the equality:

$$g_j^{-1}[h'_\sharp(g_j)] = g_j^{-1} \omega g_j^{-\sigma_\infty} \omega^{-1} \overset{R(\infty)}{=} g_j^{-1} g_j^{-\sigma_\infty} = e.$$

Thus we can replace $R(\infty)$ by $(\star)$ □

(2.5) Milnor fiber. Consider the affine hypersurface $V(C) = \{(x, y, z) \in \mathbf{C}^3; F(x, y, z) = 1\}$ where $F(X, Y, Z) = Z^d f(X/Z, Y/Z)$. The restriction of Hopf fibration to $V(C)$ is d-fold cyclic covering over $\mathbf{P}^2 - C$. Thus we have an exact sequence:

$$1 \to \pi_1(V(C)) \to \pi_1(\mathbf{P}^2 - C) \to \mathbf{Z}/d\mathbf{Z} \to 1 \tag{2.5.1}$$

Comparing with Hurewicz homomorphism, we get

Proposition (2.5.2) ([O2]). *If C is irreducible, $\pi_1(V(C))$ is isomorphic to the commutator group $\mathcal{D}(\pi_1(\mathbf{P}^2 - C))$ of $\pi_1(\mathbf{P}^2 - C)$.*

§3. **Cyclic transforms of plane curves.** Let $C \subset \mathbf{P}^2$ be a projective curve of degree d. Fixing a line at infinity L_∞, we assume that the affine curve $C^a := C \cap \mathbf{C}^2$ is defined by $f(x,y) = 0$ in $\mathbf{C}^2 = \mathbf{P}^2 - L_\infty$. We assume that $f(x,y)$ is written with mutually distinct non-zero $\alpha_1, \dots, \alpha_k$ as

$$(\sharp) \qquad f(x,y) = \prod_{i=1}^{k} (y^a - \alpha_i x^b)^{\nu_i} + (\text{lower terms}), \qquad \gcd(a,b) = 1$$

This implies that $\deg_y f(x,y) = d'$, $\deg_x f(x,y) = d''$ where $d' := a \sum_{i=1}^k \nu_i$, $d'' := b \sum_{i=1}^k \nu_i$ and $d = \max(d', d'')$ and both pencils $\{x = \eta\}_{\eta \in \mathbf{C}}$ and $\{y = \delta\}_{\delta \in \mathbf{C}}$ are admissible. Note that the assumption $(\sharp)$ does not change by the change of coordinates of the type $(x,y) \mapsto (x + \alpha, y + \beta)$.
(1) If $a = b = 1$, then $d = d' = d''$ and $L_\infty \cap C = \{[1;\alpha_i;0]; i = 1, \dots, k\}$. In particular, if $\nu_i = 1$ for each i, L_∞ is generic for C and thus L_∞ intersects transversely with C.
(2) If $a > b$ (respectively $a < b$), we have $d = d'$, $C \cap L_\infty = \{\rho_\infty := [1;0;0]\}$ (resp. $d = d''$, $C \cap L_\infty = \{\rho'_\infty := [0;1;0]\}$) and C has a singularity at ρ_∞ (resp. at ρ'_∞). The local equation at ρ_∞ (resp. ρ'_∞) takes the form:

$$(3.1.1) \qquad \begin{cases} \prod_{i=1}^k (\zeta^a - \alpha_i \xi^{a-b})^{\nu_i} + (\text{higher terms}), & \zeta = Y/X, \xi = Z/X, \ a > b \\ \prod_{i=1}^k (\zeta'^{b-a} - \alpha_i \xi'^{b})^{\nu_i} + (\text{higher terms}), & \zeta' = Z/Y, \xi' = X/Y, \ a < b \end{cases}$$

Now we consider the horizontal pencil $M_\eta = \{y = \eta\}$, $\eta \in \mathbf{C}$ and let $D = M_\beta$ be a generic pencil line. As β is generic, $D \cap C^a$ is d'' distinct points in $\mathbf{C}^2$. For an integer $n \geq 2$, we consider the n-fold cyclic covering $\varphi_n : \mathbf{C}^2 \to \mathbf{C}^2$, defined by

$$\varphi_n : \mathbf{C}^2 \to \mathbf{C}^2, \quad \varphi_n(x,y) = (x, (y - \beta)^n + \beta)$$

which is branched along D. Let $\mathcal{C}_n(C;D)^a = \varphi_n^{-1}(C^a)$ and let $\mathcal{C}_n(C;D)$ be the closure of $\mathcal{C}_n(C;D)^a$ in $\mathbf{P}^2$. To avoid the confusion, we denote the source space of φ_n by $\widetilde{\mathbf{C}^2}$ and the coordinates of $\widetilde{\mathbf{C}^2}$ by $(\tilde{x}, \tilde{y})$. Thus the line $\{\tilde{y} = \beta\}$ is equal to $\varphi_n^{-1}(D)$ and we denote it by $\widetilde{D}$. We denote the line at infinity $\mathbf{P}^2 - \widetilde{\mathbf{C}^2}$ by $\widetilde{L}_\infty$. Let $f^{(n)}(\tilde{x}, \tilde{y})$ be the defining polynomial of $\mathcal{C}_n(C;D)^a$. As $f^{(n)}(\tilde{x}, \tilde{y}) = f(\tilde{x}, (\tilde{y} - \beta)^n + \beta)$, $f^{(n)}(\tilde{x}, \tilde{y})$ takes the form:

$$(3.1.2) \qquad f^{(n)}(x,y) = \prod_{i=1}^{k} (\tilde{y}^{na} - \alpha_i \tilde{x}^b)^{\nu_i} + (\text{lower terms}).$$

Observer that $f^{(n)}(\tilde{x}, \tilde{y})$ also satisfies $(\sharp)$.

(3.2) Singularities of $\mathcal{C}_n(\mathbf{C};\mathbf{D})$. Let $\mathbf{a}_1, \dots, \mathbf{a}_s$ be the singular points of C^a and put $L_\infty \cap C = \{\mathbf{a}_\infty^1, \dots, \mathbf{a}_\infty^\ell\}$ and $\mathcal{C}_n(C;D) \cap \widetilde{L}_\infty = \{\tilde{\mathbf{a}}_\infty^i; i = 1, \dots, \tilde{\ell}\}$ where $\widetilde{L}_\infty$ is the line at infinity of the projective compactification of the source space $\widetilde{\mathbf{C}^2}$ of φ_n. Note that $\ell = k$ if $a = b = 1$ and $\ell = 1$ otherwise and $\tilde{\ell} = kb$ or 1 acoording to $na = b$ or $na \neq b$. $\mathcal{C}_n(C;D) \cap \widetilde{L}_\infty$ is either $\{[1;0;0]\}$ if $na > b$ or $\{[0;1;0]\}$ if $na < b$. It is obvious that for each $i = 1, \dots, s$, $\mathcal{C}_n(C;D)$ has n-copies of singularities $\mathbf{a}_{i,1}, \dots, \mathbf{a}_{i,n}$ which are locally isomorphic to $\mathbf{a}_i$. We denote the local Milnor number at $\mathbf{a} \in C$ by $\mu(C;\mathbf{a})$. First we recall the modified Plücker's formula for the topological Euler characteristics:

$$(3.2.1) \qquad \chi(C) = 3d - d^2 + \sum_{j=1}^{s} \mu(C;\mathbf{a}_j) + \sum_{i=1}^{\tilde{\ell}} \mu(C;\mathbf{a}_\infty^i)$$

Proposition (3.2.2). *If the branching locus D is a generic pencil line, the topological types of $(\widetilde{\mathbf{C}^2}, \mathcal{C}_n(C;D)^a)$ and $(\mathbf{P}^2, \mathcal{C}_n(C;D))$ do not depend on the choice of a generic β.*

Proof. By an easy computation, we have $\chi(\mathcal{C}_n(C;D)^a) = n(\chi(C^a) - d'') + d''$ which is independent of the choice of β. As $\chi(\mathcal{C}_n(C;D)) = \chi(\mathcal{C}_n(C;D)^a) + \tilde{\ell}$, $\chi(\mathcal{C}_n(C;D))$ is also independent of a generic β. On the other hand, the Milnor number of $\mathcal{C}_n(C;D)$ at $\mathbf{a}_{i,j}$ is equal to that of C at $\mathbf{a}_i$. Therefore by the modified Plücker's formula, the sum $\sum_{i=1}^{\tilde{\ell}} \mu(\mathcal{C}_n(C;D); \tilde{\mathbf{a}}^i_\infty)$ is also independent of β. This implies, by the upper semi-continuity of the Milnor number, the independentness of each $\mu(\mathcal{C}_n(C;D); \tilde{\mathbf{a}}^i_\infty)$. The assertion results immediately from this observation. □

If the branching line D is not generic, $\mathcal{C}_n(C;D)$ has further singularities. Let G be an arbtrary group. We denote the commutator subgroup and the center of G by $\mathcal{D}(G)$ and $\mathcal{Z}(G)$ respectively. The main result of this section is :

Theorem (3.3). *Assume that $(\sharp)$ is satisfied and D is a generic horizontal pencil line.*
(1) *The canonical homomorphism $\varphi_{n\sharp} : \pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n(C;D)^a) \to \pi_1(\mathbf{C}^2 - C^a)$ is an isomorphism.*
(2-a) *Assume $a \geq b$ (so $\deg \mathcal{C}_n(C;D) = nd$). Then there is a surjective homomorphism $\Phi_n : \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C;D)) \to \pi_1(\mathbf{P}^2 - C)$ which gives the following commutative diagram.*

$$\begin{array}{ccc} \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C;D)) & \xrightarrow{\Phi_n} & \pi_1(\mathbf{P}^2 - C) \\ \uparrow{\scriptstyle \tilde{\iota}_\sharp} & & \uparrow{\scriptstyle \iota_\sharp} \\ \pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n(C;D)^a) & \xrightarrow{\varphi_{n\sharp}} & \pi_1(\mathbf{C}^2 - C^a) \end{array}$$

where $\tilde{\iota}_\sharp$ and $\iota_\sharp$ are indeced by the respective inclusions and the kernel of Φ_n is normally generated by the class of $\omega' := \varphi_{n\sharp}^{-1}(\omega)$ where ω^{-1} is a lasso for L_∞ and ω'^{-n} is a lasso for the line at infinity $\tilde{L}_\infty$ of $\widetilde{\mathbf{C}^2}$.
(2-b) *Assume that $na \leq b$ (so $\deg \mathcal{C}_n(C;D) = \deg C^a = d$). Then we have an isomorphism: $\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C;D)) \cong \pi_1(\mathbf{P}^2 - C)$.*

Corollary (3.3.1). *Assume that $a \geq b$ and L_∞ is central for C. Then*
(1) *$\tilde{L}_\infty$ is central for $\mathcal{C}_n(C;D)$ and there is a canonical central extension of groups*

$$1 \to \mathbf{Z}/n\mathbf{Z} \xrightarrow{\iota} \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C;D)) \xrightarrow{\Phi_n} \pi_1(\mathbf{P}^2 - C) \to 1$$

(i.e., $\iota(\mathbf{Z}/n\mathbf{Z}) \subset \mathcal{Z}(\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C;D)))$) and $\mathbf{Z}/n\mathbf{Z}$ is generated by $\omega' = \varphi_{n\sharp}^{-1}(\omega)$.
(2) *The restriction of Φ_n gives an isomorphism of commutator groups*

$$\Phi_n : \mathcal{D}(\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C;D))) \to \mathcal{D}(\pi_1(\mathbf{P}^2 - C))$$

and the following exact sequences of the centers and the first homology groups:

$$\begin{array}{ccccccccc} 1 & \to & \mathbf{Z}/n\mathbf{Z} & \to & \mathcal{Z}(\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C;D))) & \xrightarrow{\Phi_n} & \mathcal{Z}(\pi_1(\mathbf{P}^2 - C)) & \to & 1 \\ 1 & \to & \mathbf{Z}/n\mathbf{Z} & \to & H_1(\mathbf{P}^2 - \mathcal{C}_n(C;D)) & \xrightarrow{\overline{\Phi}_n} & H_1(\mathbf{P}^2 - C) & \to & 1 \end{array}$$

Proof of Theorem (3.3). Taking the change of coordinates $(x,y) \mapsto (x, y+\beta)$, we may assume $D = \{y = 0\}$ for simplicity. We first prove the assertion (1). We consider the horizontal pencil

$M_\eta = \{y = \eta\}, \eta \in \mathbf{C}$. Let $\Delta_\varepsilon = \{\eta \in \mathbf{C}; |\eta| \le \varepsilon\}$, $E(\varepsilon) = \cup_{\eta\in\Delta_\varepsilon}(M_\eta^a - C^a \cap M_\eta^a)$ and $E(\varepsilon)^* = E(\varepsilon) - D$. As $M_0 = D$ is a generic pencil line, $E(\varepsilon)$ and $E(\varepsilon)^*$ are homeomorphic to the products $(M_\varepsilon - C^a \cap M_\varepsilon^a) \times \Delta_\varepsilon$ and $(M_\varepsilon - C^a \cap M_\varepsilon^a) \times \Delta_\varepsilon^*$ respectively for a sufficiently small $\varepsilon > 0$. Thus we have the isomorphism $\pi_1(E(\varepsilon)^*) = \pi_1(M_\varepsilon - C^a \cap M_0^a) \times \mathbf{Z}$ so that the canonical homomorphism $\iota_\sharp : \pi_1(M_\varepsilon - C^a \cap M_\varepsilon^a) \to \pi_1(E(\varepsilon)^*)$ is the canonical injection $g \mapsto (g, 0)$. As $\iota_\sharp : \pi_1(M_\varepsilon - C^a \cap M_\varepsilon^a) \to \pi_1(\mathbf{C}^2 - C)$ is surjective by Proposition (2.2.2), we have $\pi_1(\mathbf{C}^2 - C^a \cup D) \cong \pi_1(\mathbf{C}^2 - C^a) \times \mathbf{Z}$ where $\mathbf{Z}$ is generated by a lasso for the branch locus D and the canonical homomorphism associated with the inclusion map $a_\sharp : \pi_1(\mathbf{C}^2 - C^a \cup D) \to \pi_1(\mathbf{C}^2 - C^a)$ is the first projection under this identification. For simplicity, we denote $\mathcal{C}_n(C; D)$ by $\mathcal{C}_n(C)$ herafter. We take a lasso τ for D and fix it. We have the following exact sequence of the covering:

$$1 \to \pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n(C)^a \cup \widetilde{D}) \xrightarrow{\varphi_{n\sharp}} \pi_1(\mathbf{C}^2 - C^a \cup D) \to \mathbf{Z}/n\mathbf{Z} \to 1$$

As a subgroup of $\pi_1(\mathbf{C}^2 - C^a \cup D) \cong \pi_1(\mathbf{C}^2 - C^a) \times \mathbf{Z}$, $\pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n(C)^a \cup \widetilde{D})$ can be identified with $\pi_1(\mathbf{C}^2 - C^a) \times n\mathbf{Z}$ by $\varphi_{n\sharp}$. Note that $\varphi_{n\sharp}^{-1}(n)$ is generated by a lasso $\widetilde{\tau}$ for $\widetilde{D}$. Let us consider a subgroup $H := \varphi_{n\sharp}^{-1}(\pi_1(\mathbf{C}^2 - C^a) \times \{e\}) \subset \pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n(C)^a \cup \widetilde{D})$. Now we consider the following commutative diagram:

$$\begin{array}{ccccc} \pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n(C)^a \cup \widetilde{D}) & \supset & H & \xrightarrow{\widetilde{a}_\sharp} & \pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n(C)^a) \\ & & \downarrow \varphi_{n\sharp} & & \downarrow \varphi_{n\sharp} \\ & & \pi_1(\mathbf{C}^2 - C^a \cup D) & \xrightarrow{a_\sharp} & \pi_1(\mathbf{C}^2 - C^a) \end{array}$$

where $\widetilde{a}$ and a are respective inclusion map. As $\widetilde{a}_\sharp : \pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n(C)^a \cup \widetilde{D}) \to \pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n(C)^a)$ is surjective and $\varphi_{n\sharp}^{-1}(n\mathbf{Z})$ is included in the kernel of $\widetilde{a}_\sharp$, the restriction $\widetilde{a}_\sharp : H \to \pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n(C)^a)$ is surjective. On the other hand, as the composition $\varphi_{n\sharp} \circ \widetilde{a}_\sharp : H \to \pi_1(\mathbf{C}^2 - C^a)$ is equal to $a_\sharp \circ \varphi_{n\sharp}$, it is obviously bijective. Thus we conclude: $\widetilde{a}_\sharp : H \to \pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n(C)^a)$ and $\varphi_{n\sharp} : \pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n(C)^a) \to \pi_1(\mathbf{C}^2 - C^a)$ are isomorphisms. This proves the assertion (1).

We consider now the fundamental groups $\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C))$ and $\pi_1(\mathbf{P}^2 - C)$.
First we consider the easy case : $na \le b$ (Case (2-b)). In this case, $d = d''$, $C \cap L_\infty = \{\rho'_\infty = [0, 1, 0]\}$ and $\deg_x f(x, y) = \deg_{\tilde{x}} f^{(n)}(\tilde{x}, \tilde{y}) = d$. Take a generic horizontal pencil line $M_{\eta_0} := \{y = \eta_0\}$ with $\eta_0 \ne 0$, a base point $b_0 \in M_{\eta_0}^a$ and generators $g_1, \dots, g_d$ of $\pi_1(M_{\eta_0}^a - M_{\eta_0}^a \cap C^a; b_0)$ as before. Let $\omega = g_d \cdots g_1$. We can assume that ω is homotopic to a big circle as in Proposition (2.2.2). Take $\widetilde{\eta}_0 \in \mathbf{C}$ so that $\widetilde{\eta}_0^n = \eta_0$. We also take a base point $\widetilde{b}_0 \in \widetilde{M}_{\widetilde{\eta}_0}^a$ so that $\varphi_n(\widetilde{b}_0) = b_0$. By the definition, the pencil line $\widetilde{M}_{\widetilde{\eta}_0}$ is generic and $\varphi_n : \widetilde{M}_{\widetilde{\eta}_0}^a - \widetilde{M}_{\widetilde{\eta}_0}^a \cap \mathcal{C}_n^a(C; D) \to M_{\eta_0}^a - M_{\eta_0}^a \cap C^a$ is homeomorphism which is simply given by $(u, \widetilde{\eta}_0) \to (u, \eta_0)$. Thus we can take the pull-back $\widetilde{g}_j$ of g_j for $j = 1, \dots, d$ as generators of $\pi_1(\widetilde{M}_{\widetilde{\eta}_0}^a - \widetilde{M}_{\widetilde{\eta}_0}^a \cap \mathcal{C}_n^a(C; D))$. Let $\widetilde{\omega} = \widetilde{g}_d \cdots \widetilde{g}_1$. Then $\varphi_{n,\sharp}(\widetilde{\omega}) = \omega$. Thus the assertion (2-b) follows from

$$\begin{aligned} \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C); \widetilde{b}_0) &\cong \pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n^a(C; D); b_0)/\mathcal{N}(\widetilde{\omega}) \\ &\cong \pi_1(\mathbf{C}^2 - C^a; b_0)/\mathcal{N}(\varphi_{n,\sharp}(\widetilde{\omega})) \\ &\cong \pi_1(\mathbf{P}^2 - C; b_0) \quad \text{as } \varphi_{n,\sharp}(\widetilde{\omega}) = \omega \end{aligned}$$

where $\mathcal{N}(g)$ is the normal subgroup normally generated by g.

Now we consider the non-trivial case $a \ge b$ (Case (2-a)). Then $d = d'$ and $\deg f(x, y) = \deg_y f(x, y)$ and $nd = \deg f^{(n)}(\tilde{x}, \tilde{y}) = \deg_{\tilde{y}} f^{(n)}(\tilde{x}, \tilde{y})$. Now we consider the vertical pencil $L_\eta =$

$\{x = \eta\}$ for the computation of the monodromy relations for $\pi_1(\mathbf{C}^2 - C^a)$. Take a generic pencil line L_{η_0} and let $C^a \cap L_{\eta_0} = \{\xi_1, \dots, \xi_d\}$. Now we take $R > 0$ sufficiently large so that $C^a \cap L_{\eta_0} \subset \{\Im y > -R\}$ and $f(x, -R)$ has distinct d'' roots. We can assume that $\beta = -R$. Taking a change coordinates $(x, y) \mapsto (x, y + R)$, we may assume from the beginning that

$$D = \{y = 0\}, \quad C^a \cap L_{\eta_0} \subset \{y \in \mathbf{C}; \Im y > 0\}$$

We take the base point b_0 on the imaginary axis near the base point of the pencil B_0 as in §2 so that $\{|y| \leq |b_0|/2\} \supset C^a \cap L_{\eta_0}$ and we take a system of generators $g_1, \dots, g_d$ of $\pi_1(L^a_{\eta_0} - C^a; b_0)$ represented as $g_j = [\mathcal{L} \circ \sigma_j \circ \mathcal{L}^{-1}]$ where $\mathcal{L}$ is the segment from b_0 to $b_0/2$ and σ_j is a loop in $\{\Im y > 0\} \cap \{|y| \leq |b_0|/2\}$ starting from $b_0/2$ and $\omega = g_d \cdots g_1$ is homotopic to the big circle $\Omega : t \mapsto \exp(2\pi t i) b_0$. See the left side of Figure (3.3.A). Then by Proposition (2.2.2), we have

$$\pi_1(\mathbf{P}^2 - C) = \pi_1(\mathbf{C}^2 - C^a; b_0)/\mathcal{N}(\omega) \tag{3.3.2}$$

Now we consider the fundamental groups $\pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n(C)^a)$ and $\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C))$ using the pencil $\widetilde{L}_\eta = \{\tilde{x} = \eta\}$ in the source space $\widetilde{\mathbf{C}^2}$ of φ_n. We identify $\widetilde{L}^a_{\eta_0}$ with $\mathbf{C}$ by $\tilde{y}$-coordinate. Then by the definition of $\mathcal{C}_n(C)$, the intersection of $\mathcal{C}_n(C)^a \cap \widetilde{L}_{\eta_0}$ is n-th roots of ξ_j, for $j = 1, \dots, d$. As we have assumed $\Im \xi_j > 0$, $\mathcal{C}_n(C)^a \cap \widetilde{L}_{\eta_0}$ consists of nd points. So $\widetilde{L}_{\eta_0}$ is a generic line for $\mathcal{C}_n(C)$. Consider the conical region

$$D_j := \{(\eta_0, \tilde{y}) \in \widetilde{L}_{\eta_0}; 2\pi j/2n < \arg \tilde{y} < \pi(2j+1)/2n\}, \quad j = 0, \dots, n-1$$

is biholomorphic onto $\mathcal{H} = \{(\eta_0, y) \in L^a_{\eta_0}; \Im y > 0\}$ by φ_n. Thus the intersection $\widetilde{L}^a_{\eta_0} \cap \mathcal{C}_n(C)^a \cap D_j$ consists of d-points which correspond bijectively to those $L^a_{\eta_0} \cap C^a$.

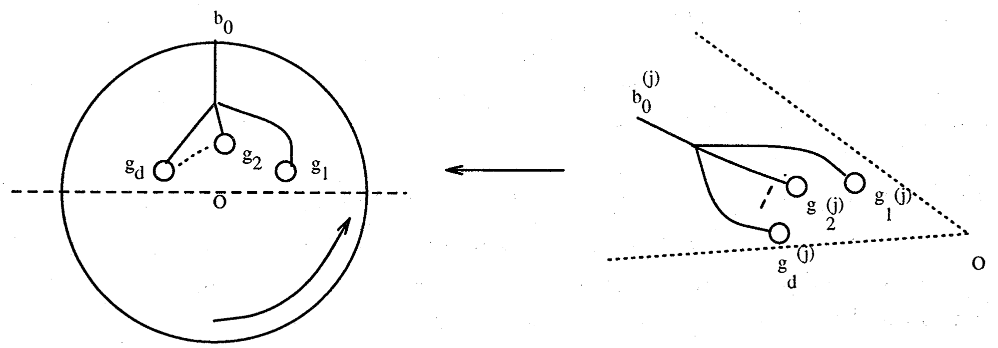


Figure (3.3.A)

Let $b_0^{(j)} \in D_j, j = 0, \dots, n-1$ be the inverse image of the base point b_0 by φ_n and we may assume $\widetilde{b_0} = b_0^{(0)}$ for example. (As a complex number, $b_0^{(j)}$ is an n-th root of b_0 for $j = 0, \dots, n-1$.) Let $\widetilde{\omega}$ be the class of the big circle: $\widetilde{\omega} : [0,1] \to \widetilde{L}^a_{\eta_0}$, $\widetilde{\omega}(t) = \widetilde{b_0} \exp(2\pi t i)$. We take the pull-back of $g_1, \dots, g_d$, $g_1^{(j)}, \dots, g_d^{(j)}$ in each conical region D_j. They gives a system of free generators of

$\pi_1(D_j - \mathcal{C}_n(C)^a \cap \widetilde{L}^a_{\eta_0}; b_0^{(j)})$. Let ℓ_j be the arc : $t \mapsto e^{it} b_0^{(0)}$, $0 \le t \le 2j\pi/n$ which connects $b_0^{(0)}$ to $b_0^{(j)}$. We associate $g_i^{(j)}$ an element $g_{i,j}$ of $\pi_1(\widetilde{L}^a_{\eta_0} - \mathcal{C}_n(C)^a \cap \widetilde{L}^a_{\eta_0}; b_0^{(0)})$ by the change of the base point: $g_i^{(j)} \mapsto g_{i,j} := \ell_j g_i^{(j)} \ell_j^{-1}$. Thus $\{g_{i,j}; 1 \le i \le d,\ 0 \le j \le n-1\}$ is a system of free generators of $\pi_1(\widetilde{L}^a_{\eta_0}; b_0^{(0)})$. See the right side of Figure (3.3.A). Let $\omega_j = g_{d,j} \cdots g_{1,j}$ for $j = 0, \dots, n-1$. Then it is easy to see that

$$\widetilde{\omega} = \omega_{n-1} \cdots \omega_0 \tag{3.3.3}$$

and by Proposition (2.2.2), we have

$$\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C); b_0^{(0)}) = \pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n(C)^a; b_0^{(0)})/\mathcal{N}(\widetilde{\omega}) \tag{3.3.4}$$

Now we examine the isomorphism: $\varphi_{n\sharp} : \pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n(C)^a; b_0^{(0)}) \to \pi_1(\mathbf{C}^2 - C^a; b_0)$ more carefully. Note first that $\varphi_n(\ell_j)$ is j-times the big circle Ω: $t \mapsto b_0 \exp(2\pi t i)$, $0 \le t \le 1$. Thus it is homotopic to ω^j. Therefore we obtain

$$\varphi_{n\sharp}(g_{i,j}) = \omega^j g_i \omega^{-j}, \quad \varphi_{n\sharp}(\omega_j) = \omega \tag{3.3.5}$$

This implies that $\omega' = \omega_1 = \cdots = \omega_n$ and

$$\varphi_{n\sharp}(\widetilde{\omega}) = \omega^n \tag{3.3.6}$$

Thus the assertion follows immediately from the isomorphisms:

$$\begin{aligned}
\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C); b_0^{(0)}) &\cong \pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n(C)^a; b_0^{(0)})/\mathcal{N}(\widetilde{\omega}) \\
&\cong \pi_1(\mathbf{C}^2 - C^a; b_0)/\mathcal{N}(\varphi_{n\sharp}(\widetilde{\omega})) \\
&\cong \pi_1(\mathbf{C}^2 - C^a; b_0)/\mathcal{N}(\omega^n)
\end{aligned}$$

By this isomorphism and (3.3.2), we have the canonical surjective homomorphism:

$$\Phi_n : \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C); b_0^{(0)}) \to \pi_1(\mathbf{P}^2 - C; b_0)$$

which is defined by $\Phi_n(g_{i,j}) = g_i$. It is obvious that Φ_n makes the diagram in (2) of Theorem (3.3) commutative. This completes the proof of Theorem (3.3). □

Proof of Corollary (3.3.1). Assume that L_∞ is central. Then $\omega \in \mathcal{Z}(\pi_1(\mathbf{C}^2 - C^a; b_0))$. As $\varphi_{n\sharp}$ is an isomorphism, $\omega' \in \mathcal{Z}(\pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n(C); b_0^{(0)}))$. Thus the normal subgroup $\mathcal{N}(\omega')$ of $\pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n(C); b_0^{(0)})$ is simply the cyclic group $\langle \omega' \rangle$ generated by ω'. We consider the Hurewicz image of ω' in $H_1(\mathbf{P}^2 - \mathcal{C}_n(C))$. Suppose that C has r irreducible components C_j of degree d_j, $j = 1, \dots, r$. Then it is obvious that $\mathcal{C}_n(C)$ consists of r irreducible components $\mathcal{C}_n(C_1), \dots, \mathcal{C}_n(C_r)$ of degree $nd_1, \dots, nd_r$ respectively. For any fixed j, d_j-elements of $\{g_{1,j}, \dots, g_{d,j}\}$ are lassos for $\mathcal{C}_n(C_j)$. Thus ω' corresponds to the class $[\omega'] = (d_1, \dots, d_r)$ of $H_1(\mathbf{P}^2 - \mathcal{C}_n(C)) \cong \mathbf{Z}^r/(nd_1, \dots, nd_r)$. Thus $[\omega']$ has order n in the first homology group. As $\omega'^n = e$ already in $\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C))$, order$(\omega') = n$ and the kernel of Φ_n is a cyclic group of order n generated by ω'. This proves the first assertion (1).

It is obvious that the image of the commutator subgroup $\mathcal{D}(\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C; D)))$ by Φ_n is surjective to the commutator subgroup $\mathcal{D}(\pi_1(\mathbf{P}^2 - C))$. On the other hand, the kernel $\mathbf{Z}/n\mathbf{Z}$ is injectively mapped to the first homology group $H_1(\mathbf{P}^2-\mathcal{C}_n(C))$. Thus $\mathcal{D}(\pi_1(\mathbf{P}^2-\mathcal{C}_n(C)))\cap\mathbf{Z}/n\mathbf{Z} = \{e\}$. Therefore Φ_n induces an isomorphism of the commutator groups. The sequence

$$1 \to \mathbf{Z}/n\mathbf{Z} \to \mathcal{Z}(\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C))) \xrightarrow{\Psi'_n} \mathcal{Z}(\pi_1(\mathbf{P}^2 - C))$$

is clearly exact. We show the surjectivity of Ψ'_n. Take $h' \in \mathcal{Z}(\pi_1(\mathbf{P}^2 - C))$ and choose $h \in \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C))$ so that $\Phi_n(h) = h'$. For any $g \in \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C))$, the image of the commutator $hgh^{-1}g^{-1}$ by Φ_n is trivial. Thus we can write $hgh^{-1}g^{-1} = \omega^a$ for some $0 \le a \le n-1$. As $[\omega]$ has order n in first homology, this implies that $a = 0$ and thus $hg = gh$ for any g. Therefore h is in the center. The last exact sequence of the assertion (2) follows by a similar argument. This completes the proof of Corollary (3.3.1). □

Remark (3.3.7). (1) We remark that the rational map $\varphi'_n : \mathbf{P}^2 \to \mathbf{P}^2$ which is associated with φ_n is defined by $\varphi'_n([X;Y;Z]) = [XZ^{n-1};Y^n;Z^n]$ and thus φ'_n is undefined at $\rho_\infty := [1;0;0] \in \mathcal{C}_n(C)$ and $\varphi'_n(\widetilde{L}_\infty - \{\rho_\infty\}) = \rho'_\infty = [0;1;0]$.
(2) In the case of $na > b > a$, there does not exist a surjective homomorphism $\Phi_n : \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C)) \to \pi_1(\mathbf{P}^2 - C)$ in general. For example, take C' a smooth curve of degree d' and let $C = \mathcal{C}_2(C;D')$ a generic two fold covering with respect to a generic line $D' := \{x = \alpha\}$. Then we take a covering $\mathcal{C}_3(C;D)$ of degree 3 with respect to a generic $D := \{y = \beta\}$. Then we know that $\deg C = 2d'$ and $\deg \mathcal{C}_3(C;D) = 3d'$ and therefore $\pi_1(\mathbf{P}^2 - \mathcal{C}_3(C;D)) = \mathbf{Z}/3d'\mathbf{Z}$ and $\pi_1(\mathbf{P}^2 - \mathcal{C}_2(C;D')) = \mathbf{Z}/2d'\mathbf{Z}$. Thus there does not exist any surjective homomorphism.

(3.4) Generic cyclic covering. Now we consider the generic case:

$$f(x,y) = \prod_{i=1}^{d}(y - \alpha_i x) + (\text{lower terms}), \quad \alpha_1,\dots,\alpha_d \in \mathbf{C}^* \tag{3.4.1}$$

This is always the case if we choose the line at infinity L_∞ to be generic and then generic affine coordinates (x,y). Take positive integers $n \ge m \ge 1$ and we denote $\mathcal{C}_n(C;D)$ by $\mathcal{C}_n(C)$ and $\mathcal{C}_m(\mathcal{C}_n(C;D);D')$ by $\mathcal{C}_{m,n}(C)$ where $D = \{y = \beta\}$ and $D' = \{x = \alpha\}$ with generic α,β. Note that $\mathcal{C}_n(C) = \mathcal{C}_{1,n}(C)$. The topology of the complement of $\mathcal{C}_{m,n}(C)$ depends only on C and m,n. We will refer $\mathcal{C}_n(C)$ and $\mathcal{C}_{m,n}(C)$ as *a generic n-fold* (respectively *a generic (m,n)-fold*) *covering transform* of C. They are defined in $\mathbf{C}^2$ by

$$\mathcal{C}_n(C)^a = \{(\tilde{x},\tilde{y}) \in \mathbf{C}^2; f(\tilde{x},\tilde{y}^n) = 0\}, \quad \mathcal{C}_{m,n}(C)^a = \{(\tilde{x},\tilde{y}) \in \mathbf{C}^2; f(\tilde{x}^m,\tilde{y}^n) = 0\}$$

taking a change of coordinate $(x,y) \mapsto (x+\alpha, y+\beta)$ if necessary. If $n > m$, $\mathcal{C}_{m,n}(C)$ has one singularity at $\rho_\infty = [1;0;0]$ and the local equation takes the following form:

$$\prod_{i=1}^{d}(\zeta^n - \alpha_i \xi^{n-m}) + (\text{higher terms}), \quad \zeta = Y/X, \xi = Z/X$$

Therefore $\mathcal{C}_{m,n}(C)$ is locally $d\ \gcd(m,n)$ irreducible components at $\mathbf{a}_\infty$. $(\mathcal{C}_{m,n}(C),\rho_\infty)$ is topologically equivalent to the germ of a Brieskorn singularity $B((n-m)d, nd)$ where $B(p,q) := \{\xi^p - \zeta^q\} = 0$. In the case $m = n$, we have no singularity at infinity. By Theorem (3.3) and Corollary (3.3.1), we have the following.

Theorem (3.5). *Let $\mathcal{C}_n(C)$ and $\mathcal{C}_{m,n}(C)$ be as above. Then the canonical homomorphisms*

$$\pi_1(\widetilde{\widetilde{\mathbf{C}^2}} - \mathcal{C}_{m,n}(C)^a) \xrightarrow{\varphi_{m\sharp}} \pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n(C)^a) \xrightarrow{\varphi_{n\sharp}} \pi_1(\mathbf{C}^2 - C^a)$$

and $\Phi_m : \pi_1(\mathbf{P}^2 - \mathcal{C}_{m,n}(C)) \to \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C))$ *are isomorphisms. There exxist canonical central extensions of groups*

$$\begin{array}{ccccccccc} 1 & \to & \mathbf{Z}/n\mathbf{Z} & \xrightarrow{\iota} & \pi_1(\mathbf{P}^2 - \mathcal{C}_{m,n}(C)) & \xrightarrow{\Phi_{m,n}} & \pi_1(\mathbf{P}^2 - C) & \to & 1 \\ & & \big\downarrow \mathrm{id} & \circlearrowleft & \cong \big\downarrow \Phi_m & \circlearrowleft & \big\downarrow \mathrm{id} & & \\ 1 & \to & \mathbf{Z}/n\mathbf{Z} & \xrightarrow{\iota'} & \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C)) & \xrightarrow{\Phi_n} & \pi_1(\mathbf{P}^2 - C) & \to & 1 \end{array}$$

The kernel $\operatorname{Ker}\Phi_n$ *(respectively* $\operatorname{Ker}\Phi_{m,n}$*) is generated by an element* ω' *(resp.* $\omega'' = \Phi_m^{-1}(\omega')$*) in the center such that* ω'^n *(resp.* ω''^n*) is a lasso for* $\widetilde{L}_\infty$ *(resp. for* $\widetilde{\widetilde{L}}_\infty$*). The restriction of* $\Phi_{m,n}$*,* Φ_m *and* Φ_n *give an isomorphism of the respective commutator groups*

$$\Phi_{m,n,\mathcal{D}} : \mathcal{D}(\pi_1(\mathbf{P}^2 - \mathcal{C}_{m,n}(C))) \xrightarrow{\Phi_{m,\mathcal{D}}} \mathcal{D}(\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C))) \xrightarrow{\Phi_{n,\mathcal{D}}} \mathcal{D}(\pi_1(\mathbf{P}^2 - C))$$

and exact sequences of the centers and the first homology groups:

$$\begin{array}{ccccccccc} 1 & \to & \mathbf{Z}/n\mathbf{Z} & \to & \mathcal{Z}(\pi_1(\mathbf{P}^2 - \mathcal{C}_{m,n}(C))) & \xrightarrow{\Phi_{m,n}} & \mathcal{Z}(\pi_1(\mathbf{P}^2 - C)) & \to & 1 \\ 1 & \to & \mathbf{Z}/n\mathbf{Z} & \to & H_1(\mathbf{P}^2 - \mathcal{C}_{m,n}(C)) & \xrightarrow{\overline{\Phi}_{m,n}} & H_1(\mathbf{P}^2 - C) & \to & 1 \end{array}$$

Let $\{\mathbf{a}_1, \dots, \mathbf{a}_s\}$ be singular points as before. Then $\mathcal{C}_n(C)$ (respectively $\mathcal{C}_{m,n}(C)$) has n copies (resp. nm copies) of $\mathbf{a}_i$ for each $i = 1, \dots, s$ and one singularity at $\rho_\infty := [1; 0; 0]$ except the case $n = m$. The curve $\mathcal{C}_{n,n}(C)$ has no singularity at infinity. The similar assertion for $\mathcal{C}_{n,n}(C)$ is obtained independently by Shimada [Sh]. By Corollary (3.3.1), we have the following.

Corollary (3.5.1). (1) $\pi_1(\mathbf{P}^2 - \mathcal{C}_{m,n}(C))$ *is abelian if and only if* $\pi_1(\mathbf{P}^2 - C)$ *is abelian.*
(2) *Assume that* C *is irreducible. Then the fundamental groups* $\pi_1(V(\mathcal{C}_{m,n}(C)))$ *and* $\pi_1(V(C))$ *of the respective Milnor fibers* $V(\mathcal{C}_{m,n}(C))$ *of* $\mathcal{C}_{m,n}(C)$ *and* $V(C)$ *of* C *are isomorphic.*

Proof. The assertion (1) follows from (2) of Corollary (3.3.1). The assertion (2) is immediate from Proposition (2.5.2) and Corollary (3.3.1). □

The following is immediate consequence of Corollary (3.3.1) and Corollary (2.4.2).

Corollary (3.5.2). $\widetilde{\widetilde{L}}_\infty$ *is central for* $\mathcal{C}_{m,n}(C)$ *i.e.,* $\pi_1(\mathbf{P}^2 - \mathcal{C}_{m,n}(C) \cup \widetilde{\widetilde{L}}_\infty)$ *is isomorphic to the fundamental group of the generic affine complement of* $\mathcal{C}_{m,n}(C)$.

First we consider the following condition for a group G:

(H.I.C) $$\mathcal{Z}(G) \cap \mathcal{D}(G) = \{e\}$$

This is equivalent to the injectivity of the composition: $\mathcal{Z}(G) \hookrightarrow G \to H_1(G) := G/\mathcal{D}(G)$. When this condition is satisfied, we say that G satisfies *homological injectivity condition of the center* (or (H.I.C)-condition in short).

Corollary (3.5.3). *Let* $C = C_1 \cup \cdots \cup C_r$ *and* $C' = C'_1 \cup \cdots \cup C'_r$ *be projective curves with same number of irreducible components and assume that* $\operatorname{degree}(C_i) = \operatorname{degree}(C'_i) = d_i$ *for* $i = 1, \dots, r$

and assume that $\pi_1(\mathbf{P}^2 - C')$ satisfies (H.I.C)-condition. Assume that $\pi_1(\mathbf{P}^2 - \mathcal{C}_{m,n}(C))$ and $\pi_1(\mathbf{P}^2 - \mathcal{C}_{m,n}(C'))$ are isomorphic. Then $\pi_1(\mathbf{P}^2 - C)$ and $\pi_1(\mathbf{P}^2 - C')$ are isomorphic.

Proof. We may assume that $m = 1$ by Theorem (3.3). Suppose that $\alpha : \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C)) \to \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C)')$ is an isomorphism. This induces isomorphisms of the respective commutator subgroups, centers and the first homology groups. We consider the exact sequences given by Corollary (3.3.1):

$$\begin{array}{ccccccccc} 1 & \to & \mathbf{Z}/n\mathbf{Z} & \to & \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C)) & \xrightarrow{\Phi_n} & \pi_1(\mathbf{P}^2 - C) & \to & 1 \\ & & & & \downarrow \alpha & & & & \\ 1 & \to & \mathbf{Z}/n\mathbf{Z} & \to & \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C')) & \xrightarrow{\Phi'_n} & \pi_1(\mathbf{P}^2 - C') & \to & 1 \end{array}$$

Let ω' and ω'' be the generator of the kernels of Φ_n and Φ'_n respectively. As $[\omega'] = [(d_1, \dots, d_r)] \in H_1(\mathbf{P}^2 - \mathcal{C}_n(C)) = \mathbf{Z}^r/(nd_1, \dots, nd_r)$ in the notation of (2.1) and $[\omega']$ has order n, the homology class $[\alpha(\omega')]$ corresponding to $\alpha(\omega')$ has also order n in $H_1(\mathbf{P}^2 - \mathcal{C}_n(C'))$, thus $[\alpha(\omega')]$ is also anihilated by n. Therefore it is homologous to $[(ad_1, \dots, ad_r)] \in H_1(\mathbf{P}^2 - \mathcal{C}_n(C'))$ for some $a \in \mathbf{Z}$. This implies $[\Phi'_n(\alpha(\omega'))] = 0 \in H_1(\mathbf{P}^2 - C')$ and therefore, by (3) of Theorem (3.3), that $\Phi'_n(\alpha(\omega')) \in \mathcal{D}(\pi_1(\mathbf{P}^2 - C'))$. Therefore $\Phi'_n(\alpha(\omega')) \in \mathcal{D}(\pi_1(\mathbf{P}^2 - C')) \cap \mathcal{Z}(\pi_1(\mathbf{P}^2 - C'))$. By the (H.I.C)-condition, this implies that $\Phi'_n(\alpha(\omega')) = e$. Thus by the above exact sequence, $\alpha(\omega') = (\omega'')^\beta$ for some $\beta \in \mathbf{N}$ with $\gcd(\beta, n) = 1$. Thus the restriction of α to $\mathrm{Ker}(\Phi_n) \cong \mathbf{Z}/n\mathbf{Z}$ is an isomorphism on to $\mathrm{Ker}(\Phi'_n) \cong \mathbf{Z}/n\mathbf{Z}$. Thus it induces an isomorphism : $\bar\alpha : \pi_1(\mathbf{P}^2 - C) \to \pi_1(\mathbf{P}^2 - C')$. □

Remark (3.6). (1) Take a non-generic line $D = \{y = \beta\}$ for C and consider the corresponding cyclic covering branched along D, $\varphi_n : \mathbf{C}^2 \to \mathbf{C}^2$. Then the assertions in Theorem (3.3) and Corollary (3.3.1) for the pull back $C' = \varphi_n^{-1}(C)$ may fail in general. For example, we can take the quartic defined by (5.1.1) in §5. Then L_∞ is central for C and $\pi_1(\mathbf{P}^2 - C) = \mathbf{Z}/4\mathbf{Z}$. Take $D = \{y = 0\}$ and consider $\varphi_2 : \mathbf{C}^2 \to \mathbf{C}^2$, $\varphi_2(x, y) = (x, y^2)$. Then the pull back Z_4 of C is a so called Zariski's three cuspidal quartic and $\pi_1(\mathbf{P}^2 - Z_4)$ ia a finite non-abelian group of order 12 ([Z1],[O5]). See also §5.
(2) We do not have any example of a plane curve C such that $\pi_1(\mathbf{P}^2 - C)$ does not satisfy the (H.I.C)-condition.

§5. Zariski's quartic and Zariski pairs.

In this section, we apply the results of §3 and §4 to construct plane curves whose complement have interesting fundamental groups.

(5.1) Zariski's three cuspidal quartics. Let Z_4 be an irreducible quartic with three cusps. Such a curve is a rational curve. For example, we can take the following curve which is defined in $\mathbf{C}^2$ by the following equation ([O6]):

(5.1.1) $$Z_4^a = \{(x, y) \in \mathbf{C}^2; (x-1)^3(3x+5) - 6(x-1)^2(y^2-1) - (y^2-1)^2 = 0\}$$

We call such a curve *a Zariski's three cuspidal quartic*. It is known that the fundamental group $\pi_1(\mathbf{C}^2 - Z_4)$ and $\pi_1(\mathbf{P}^2 - Z_4)$ have the following representations ([Z1],[O6]):

(5.1.2) $$\begin{cases} \pi_1(\mathbf{C}^2 - Z_4) & = \langle \rho, \xi; \{\rho, \xi\} = e,\ \rho^2 = \xi^2 \rangle \\ \pi_1(\mathbf{P}^2 - Z_4) & = \langle \rho, \xi; \{\rho, \xi\} = e,\ \rho^2 = \xi^2, \rho^4 = e \rangle \end{cases}$$

where ρ and ξ are lassos for C and $\{\rho, \xi\} := \rho\xi\rho\xi^{-1}\rho^{-1}\xi^{-1}$. The relation $\{\rho, \xi\} = e$ is equivalent to $\rho\xi\rho = \xi\rho\xi$. The element ω is given by $\rho^2\xi^2 (= \rho^4)$. Recall that ω^{-1} is a lasso for L_∞ and is contained in the center. A Zariski's three cuspidal quartic is the first example whose complement has a non-abelian finite fundamental group. We first recall the proof of the finiteness.

Lemma (5.1.3) ([Z1]). *Put*

$$G_1 = \langle \rho, \xi; \{\rho, \xi\} = e,\ \rho^2 = \xi^2, \rho^4 = e\rangle.$$

Then G_1 is a finite group of order 12 such that $\mathcal{D}(G_1) = \langle \rho^2\xi\rho\rangle \cong \mathbf{Z}/3\mathbf{Z}$, $\mathcal{Z}(G_1) = \langle \rho^2\rangle \cong \mathbf{Z}/2\mathbf{Z}$ and $H_1(G_1) \cong \mathbf{Z}/4\mathbf{Z}$ and it is generated by the class of ρ

(5.2) Generic transforms of a Zariski's quartic. Let $\mathcal{C}_n(Z_4)$ (respectively $\mathcal{C}_{n,n}(Z_4)$) be a generic cyclic transform of degree n (resp. of (n,n)) of the Zariski's quartic Z_4 and let $\mathcal{J}_n(Z_4)$ be a generic Jung transform of degree n of the Zariski's quartic Z_4. The singularities of $\mathcal{C}_n(Z_4)$ (respectively of $\mathcal{C}_{n,n}(Z_4)$) are $3n$ cusps (resp. $3n^2$ cusps). $\mathcal{C}_n(Z_4)$ has one more singularity at $\rho_\infty \in L_\infty$ and $(\mathcal{C}_n(Z_4), \rho_\infty)$ is equal to $B((n-1)d, nd) := \{\zeta^{nd} - \xi^{d(n-1)}\} = 0\}$. On the other hand, $\mathcal{J}_n(Z_4)$ is a rational curve which has 3 cusps and one more singularity at infinity $\rho_\infty \in \mathcal{J}_n(Z_4) \cap L_\infty$. $(\mathcal{J}_n(Z_4), \rho_\infty)$ is topologically equal to $B(n-1, n; d) := \{(\xi^{n-1} + \zeta^n)^d - (\zeta\xi^{n-1})^d = 0\}$. By Theorem (3.5) and Theorem (4.3), we have the following:

Theorem (5.3). *The affine fundamental groups $\pi_1(\mathbf{C}^2 - \mathcal{C}_n(Z_4)^a)$, $\pi_1(\mathbf{C}^2 - \mathcal{J}_n(Z_4)^a)$ are isomorphic to $\pi_1(\mathbf{C}^2 - Z_4) \cong \langle \rho_n, \xi_n; \{\rho_n, \xi_n\} = e,\ \rho_n^2 = \xi_n^2\rangle$.*
(1) The projective fundamental groups $\pi_1(\mathbf{P}^2 - \mathcal{C}_n(Z_4))$ and $\pi_1(\mathbf{P}^2 - \mathcal{J}_n(Z_4))$ are isomorphic to G_n where G_n is defined by $G_n := \langle \rho_n, \xi_n; \{\rho_n, \xi_n\} = e,\ \rho_n^2 = \xi_n^2, \rho_n^{4n} = e\rangle$. Moreover we have a central extension of groups:

$$1 \to \mathbf{Z}/n\mathbf{Z} \to G_n \xrightarrow{\Phi_n} G_1 \to 1 \tag{5.3.1}$$

defined by $\Phi_n(\rho_n) = \rho$ and $\Phi_n(\xi_n) = \xi$ and $\operatorname{Ker}\Phi_n$ is generated by ρ_n^4. In particular, we have $|G_n| = 12n$, $\mathcal{D}(G_n) = \langle \beta_n\rangle \cong \mathbf{Z}/3\mathbf{Z}$ where $\beta_n = [\rho_n, \xi_n]$ and $\mathcal{Z}(G_n) = \langle \rho_n^2\rangle \cong \mathbf{Z}/2n\mathbf{Z}$.
(2) The Hurewicz sequence $1 \to \mathcal{D}(G_n) \to G_n \to H_1(G_n) \to 1$ has a canonical cross section $\theta : H_1(G_n) \to G_n$ which is given by $\theta(\bar\rho_n) = \rho_n$. This gives G_n a structure of semi-direct product $\mathbf{Z}/3$ and $\mathbf{Z}/4n\mathbf{Z}$ which is determined by $\rho_n\beta_n\rho_n^{-1} = \beta_n^2$.
(3) G_n is identified with the subgroup of the permutation group $\mathfrak{S}_{12n}$ of $12n$ elements $\{x_i, y_j, z_k; 1 \le i, j, k \le 4n\}$ generated by two permutations: $\sigma_n = (x_1, \dots, x_{4n})(y_1, \dots, y_{4n})(z_1, \dots, z_{4n})$ and $\tau_n = (x_1, y_1, x_3, y_3, \dots, x_{4n-1}, y_{4n-1})(x_2, z_1, x_4, z_3, \dots, x_{4n}, z_{4n-1})(y_2, z_2, y_4, z_4, \dots, y_{4n}, z_{4n})$.

(5.4) Zariski pairs. Let C and C' be plane curves of the same degree and let $\Sigma(C) = \{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ and $\Sigma(C') = \{\mathbf{a}'_1, \dots, \mathbf{a}'_{m'}\}$ be the singular points of C and C' respectively. Assume that L_∞ is generic for both of them. We say that $\{C, C'\}$ is a *Zariski pair* if (1) $m = m'$ and the germ of the singularity $(C, \mathbf{a}_j)$ is topologically equivalent to $(C', \mathbf{a}'_j)$ for each j and (2) there exist neighborhoods $N(C)$ and $N(C')$ of C and C' respectively so that $(N(C), C)$ and $(N(C'), C')$ are homeomorphic and (3) the pair $(\mathbf{P}^2, C)$ is not homeomorphic to the pair $(\mathbf{P}^2, C')$ ([Ba]).

The assumption (2)is not necessary if C and C' are irreducible. For our purpose, we replace (3) by one of the following:
(Z-1) $\pi_1(\mathbf{P}^2 - C) \not\cong \pi_1(\mathbf{P}^2 - C')$,
(Z-2) $\pi_1(\mathbf{C}^2 - C^a) \not\cong \pi_1(\mathbf{C}^2 - C'^a)$, where $\mathbf{C}^2 = \mathbf{P}^2 - L_\infty$ and L_∞ is generic,
(Z-3) $\mathcal{D}(\pi_1(\mathbf{P}^2 - C)) \not\cong \mathcal{D}(\pi_1(\mathbf{P}^2 - C'))$.

We say that $\{C, C'\}$ is a *strong Zariski pair* if the conditions (1), (2) and the condition (Z-1) are satisfied. Similarly we say $\{C, C'\}$ is a *strong generic affine Zariski pair* (respectively *strong Milnor pair*) if the conditions (1), (2) and the condition (Z-2) (resp. (Z-3)) are satisfied.

If C and C' are irreducible curves satisfying (1) and (2), $\{C, C'\}$ is a strong Milnor pair if and only if the fundamental groups of the respective Milnor fibers $V(C)$ and $V(C')$ are not isomorphic by Proposition (2.5.2). The above three conditions (Z-1)$\sim$ (Z-3) are related by the following.

Proposition (5.4.1). (1) *If $\{C, C'\}$ is a strong Milnor pair, $\{C, C'\}$ is a strong Zariski pair as well as a strong generic affine Zariski pair.*
(2) *Assume that C and C' are irreducible and assume that $\{C, C'\}$ is a strong Zariski pair and either $\pi_1(\mathbf{C}^2 - C^a)$ or $\pi_1(\mathbf{C}^2 - C'^a)$ satisfies* (H.I)*-condition. Then $\{C, C'\}$ is a strong generic affine Zariski pair.*

The results of §3,4 can be restated as follows.

Theorem (5.5). *Let C, C' be projective curves and let $\mathcal{C}_{n,m}(C), \mathcal{C}_{n,m}(C')$ (respectively $\mathcal{J}_n(C)$ and $\mathcal{J}_n(C')$) be the generic (n,m)-fold cyclic transforms (resp. generic Jung transform of degree n) of C and C' respectively.*
(1) *Assume that $\{C, C'\}$ is a strong affine Zariski pair (respectively strong Milnor pair). Then $\{\mathcal{C}_{n,m}(C), \mathcal{C}_{n,m}(C')\}$ is a strong affine Zariski pair (resp. strong Milnor pair).*
(2) *Assume that $\{C, C'\}$ is a strong Zariski pair. We assume also either C or C' satisfies (H.I)-condition. Then $\{\mathcal{C}_{n,m}(C), \mathcal{C}_{n,m}(C')\}$ is a strong Zariski pair.*

The same assertion holds for $\mathcal{J}_n(C)$ and $\mathcal{J}_n(C')$.

Example (5.6) (A new example of a Zariski pair). We apply generic 2-covering or $(2,2)$-covering and generic Jung transform of degree 2 to the pair $\{Z_6, Z_6'\}$ to obtain three strong Zariski pairs of curves of degree 12:
(1) Take $\{\mathcal{C}_2(Z_6), \mathcal{C}_2(Z_6')\}$. Both curves have 12 cusps ($= B(2,3)$) and one $B(6,12)$ singularity at infinity. $\pi_1(\mathbf{P}^2 - \mathcal{C}_2(Z_6))$ is a central $\mathbf{Z}/2\mathbf{Z}$-extension of $\mathbf{Z}/3\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$ and it is denoted by $G(3;2;4)$ in [O5]. $\pi_1(\mathbf{P}^2 - \mathcal{C}_2(Z_6'))$ is isomorphic to a cyclic group $\mathbf{Z}/12\mathbf{Z}$.
(2) Take $\{\mathcal{C}_{2,2}(Z_6), \mathcal{C}_{2,2}(Z_6')\}$. They have 24 cusps. The fundamental groups are as above.
(3) Take $\{\mathcal{J}_2(Z_6), \mathcal{J}_2(Z_6')\}$. Singularities are 6 cusps and one $B(6,18)$. The fundamental groups are as in (1).
(4) Take $\{\mathcal{C}_2(\mathcal{J}_2(Z_6)), \mathcal{C}_2(\mathcal{J}_2(Z_6'))\}$. Singularities are 12 cusps and two $B(6,6)$ singularities.

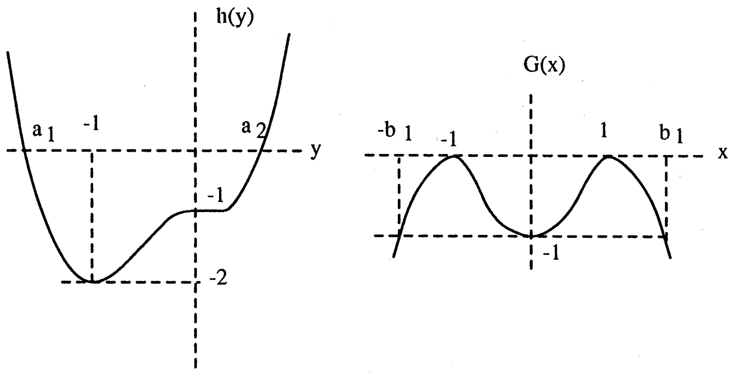
(5) We now propose a new strong Zariski pair $\{C_1, C_2\}$ of degree 12. First for C_1, we take the generic cyclic transform $\mathcal{C}_3(Z_4)$ of degree 3 of a Zariski's three cuspidal quartic. Recall that C_1 has 9 cusps and one $B(8,12)$ singularity at $\rho_\infty := [1;0;0]$. We have seen that $\pi_1(\mathbf{P}^2 - C_1)$ is G_3, a finite group of order 36. We will construct below another irreducible curve C_2 of degree 12 with 9 cusps and one $B(8,12)$ singularity at ρ_∞ such that $\pi_1(\mathbf{P}^2 - C_2) \cong G(3;2;4)$ where $G(3;2;4)$ is introduced in [O5] (see also §6) and it is a central extension of $\mathbf{Z}/3\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$ by $\mathbf{Z}/2\mathbf{Z}$.
(6) Take $\{\mathcal{C}_{3,3}(Z_4), \mathcal{C}_3(C_2; D)\}$ where $D = \{x = \alpha\}$ is generic. They are curves of degree 12 with 27 cusps. The fundamental groups $\pi_1(\mathbf{P}^2 - \mathcal{C}_{3,3}(Z_4))$ and $\pi_1(\mathbf{P}^2 - \mathcal{C}_3(C_2; D))$ are isomorphic to the case (5).

Construction of $\mathbf{C}_2$. Let us consider a family of affine curves $K^a(\tau) = \{(x,y) \in \mathbf{C}^2; h(y)^3 = \tau G(x)\}$ $(\tau \in \mathbf{C}^*)$ where $h(y) = 3y^4 + 4y^3 - 1$, $G(x) = -(x^2-1)^2$.

Figure (5.6.A)

Let $K(\tau)$ be the projective compactification of $K^a(\tau)$. Let $a_1, \dots, a_4$ be the solution of $h(y) = 0$. Here we assume that a_1, a_2 are real roots with $a_1 < a_2$ and $a_3 = \overline{a_4}$. By a direct computation, we see that $K(\tau)$ has 8 cusp singularities at $\{A_1, A_1', \dots, A_4, A_4'\}$ where $A_i := (1, a_i)$, $A_i' := (-1, a_i)$ for $i = 1, \dots, 4$ and a $B(8,12)$ singularity at $\rho_\infty = [1;0;0]$. Putting $\tau = 1$, $K(1)$ has one more cusp at $A_0 := (-1, 0)$. For C_2, we take $K(1)$. As $\pi_1(\mathbf{P}^2 - K(\tau)) = G(3;2;4)$ by [O5][1], $\pi_1(\mathbf{P}^2 - C_2)$ is not

[1]In [O5], we have only considered the curves of type $f(y) = g(x)$ with $\deg f = \deg g$. However the same



smaller than $G(3;2;4)$ as there exists a surjective morphism from $\pi_1(\mathbf{P}^2 - K(1))$ to $\pi_1(\mathbf{P}^2 - K(\tau)) = G(3;2;4)$. In fact, we assert that $\pi_1(\mathbf{P}^2 - C_2) = G(3;2;4)$.

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assertion holds if $\deg f(y) \geq \deg g(x)$.

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